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Variations on the planar Landau problem: canonical transformations, a purely linear potential and the half-plane

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Abstract
The ordinary Landau problem of a charged particle in a plane subjected to a perpendicular homogeneous and static magnetic field is reconsidered from different points of view. The role of phase space canonical transformations and their relation to a choice of gauge in the solution of the problem is addressed. The Landau problem is then extended to different contexts, in particular the singular situation of a purely linear potential term being added as an interaction, for which a complete purely algebraic solution is presented. This solution is then exploited to solve this same singular Landau problem in the half-plane, with as motivation the potential relevance of such a geometry for quantum Hall measurements in the presence of an electric field or a gravitational quantum well.

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1. Introduction

The classic textbook example [1] of the quantum Landau problem has remained a constant source of fascination and inspiration [2], in fields apparently so diverse as two-dimensional collective quantum fermionic systems [3, 4], the search towards a fundamental unification of gravity with the other quantum interactions, or noncommutative deformation quantization

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of geometries [5, 6]. The same algebraic structures are also realized in M-theory in specific limits of some background field configurations [7]. It is in view of the latter developments as well as the phenomenology of the integer and fractional quantum Hall effects that in recent years the Landau problem has become once again the focus of intense interest.

Yet, there remain somewhat intriguing issues open even for the simple original Landau problem. Consider thus a charged particle of mass m moving in an Euclidean plane of coordinates (x_1, x_2) and subjected to a static and homogeneous magnetic field perpendicular to that plane, with a component B along the right-handed perpendicular direction which, without loss of generality (by choosing the plane orientation appropriately), may be taken to be positive, $B > 0$ (this factor, B, is also normalized so as to absorb the charge of the particle). Denoting by $(A_1(x_1, x_2), A_2(x_1, x_2))$ the components of a vector potential from which the magnetic field derives, $\partial_1 A_2 - \partial_2 A_1 = B$, it is well known that the dynamics of the system is specified through the variational principle from the following Lagrange function,

$$L = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \dot{x}_1 A_1(x_1, x_2) + \dot{x}_2 A_2(x_1, x_2), \tag{1}$$

with a Hamiltonian function for the canonically conjugate pairs of phase space variables (x_i, p_i) $(i = 1, 2)$,

$$H = \frac{1}{2m}(p_1 - A_1(x_1, x_2))^2 + \frac{1}{2m}(p_2 - A_2(x_1, x_2))^2. \tag{2}$$

The usual discussion [1] considers the Landau gauge for the vector potential,

$$A_1^{\mathrm{Landau}} = -Bx_2, \qquad A_2^{\mathrm{Landau}} = 0, \tag{3}$$

in which case one has

$$H = \frac{1}{2m}p_2^2 + \frac{1}{2}m\omega_c^2\left(x_2 + \frac{1}{B}p_1\right)^2, \tag{4}$$

with the cyclotron frequency $\omega_c = B/m$. For the quantized system, by introducing the Fock algebra generators

$$a = \sqrt{\frac{m\omega_c}{2\hbar}}\left[\left(\hat{x}_2 + \frac{1}{B}\hat{p}_1\right) + \frac{\mathrm{i}}{m\omega_c}\hat{p}_2\right], \qquad a^\dagger = \sqrt{\frac{m\omega_c}{2\hbar}}\left[\left(\hat{x}_2 + \frac{1}{B}\hat{p}_1\right) - \frac{\mathrm{i}}{m\omega_c}\hat{p}_2\right], \tag{5}$$

with

$$[\hat{x}_1, \hat{p}_1] = \mathrm{i}\hbar\mathbb{I}, \qquad [a, \hat{p}_1] = 0, \qquad [a, \hat{x}_1] = -\mathrm{i}\sqrt{\frac{\hbar}{2B}}\mathbb{I}, \qquad [a, a^\dagger] = \mathbb{I}, \tag{6}$$

such that $\hat{H} = \hbar\omega_c(a^\dagger a + 1/2)$, it is clear that the energy spectrum is spanned by Fock states $|n, p_1\rangle$ $(n = 0, 1, 2, \ldots)$ with an energy $\hbar\omega_c(n + 1/2)$ which is degenerate in p_1—the famous Landau levels—the latter real variable p_1 labelling the $\hat{p}_1$ eigenstates.

However what is puzzling, perhaps, about this solution is the fact that because of the free particle plane wave component of the configuration space wave function representation of these states related to the p_1 eigenvalue, none of these states is normalizable,

$$\langle n, p_1 | m, p_1'\rangle = \delta_{nm}\,\delta(p_1 - p_1'), \tag{7}$$

while this basis of states is non-countable and their wave functions are localized only in the x_2 direction (through the Gaussian factor and Hermite polynomials in the $(x_2 + p_1/B)$ variable) but not at all in the x_1 direction where they display complete delocalization (note also that the $(\hat{x}_1, \hat{p}_1)$ sector does not commute with the Fock algebra, only the conjugate momentum operator, $\hat{p}_1$, does). And yet the classical trajectories of such a particle are circles of which the radius is a function of the energy of the solution, the angular frequency is ω_c and the magnetic centre is pinned at a static position in the plane dependent on the initial conditions. Hence, rather than the above quantum states, one should expect that there ought to exist another basis of the energy eigenstates which describes normalizable and localized wave functions.

Indeed as is well known, in the circular or symmetric gauge,

$$A_1^{\rm sym} = -\tfrac{1}{2}Bx_2, \qquad A_2^{\rm sym} = +\tfrac{1}{2}Bx_1, \tag{8}$$

once expanded, the Hamiltonian

$$H = \frac{1}{2m}\left(p_1 + \frac{1}{2}Bx_2\right)^2 + \frac{1}{2m}\left(p_2 - \frac{1}{2}Bx_1\right)^2, \tag{9}$$

coincides with that of a two-dimensional spherically symmetric harmonic oscillator of angular frequency $\omega_c/2$ to which a term proportional to its angular momentum is added. Working in a complex parametrization of the plane, it is clear[5] that the system is then diagonalized with a countable energy eigenspectrum of Fock states, possessing the same energy spectrum as above, but now represented by wave functions which are all localized and normalizable (and in fact centred onto the point $(x_1, x_2) = (0, 0)$).

At first sight, what distinguishes the above two gauge choices at the quantum level is a redefinition of the wave functions of quantum states by a pure phase factor, $e^{i\chi(x_1,x_2)}$, related to the gauge transformation mapping the two choices of vector potentials into one another,

$$A_i^{\rm sym} = A_i^{\rm Landau} + \partial_i\chi, \qquad \chi(x_1, x_2) = \tfrac{1}{2}Bx_1x_2. \tag{10}$$

The phase factor $e^{i\chi}$ being singular at the point at infinity in the plane could be thought to be the reason for the non-normalizability and non-localizability of energy eigenstates in the Landau gauge. However, being a pure phase, such a phase redefinition alone cannot explain why out of a localized and normalized wave function in the symmetric gauge, one obtains a delocalized and non-normalizable one in the Landau gauge.

In section 2, this question is addressed in detail and resolved. Then, in section 3, using the understanding gained from section 2, and mostly to set notations for later use, the original Landau problem is extended by adding an interaction potential energy which combines that of a spherically symmetric harmonic well and a linear potential. When the harmonic well is removed an apparent puzzle arises, the resolution of which is discussed in section 4 using a purely algebraic approach. To the best of our knowledge, the present algebraic solution—as opposed to a wave function solution of the Schrödinger equation in the Landau gauge—for the Landau problem extended with a linear potential is not available in the literature. Finally, using the insight provided by this construction and motivated by some physical considerations, section 5 discusses the same linearly extended Landau problem in the half-plane. The paper ends with some conclusions.

2. The ordinary Landau problem

2.1. A general choice of gauge

With the Lagrangian defined in (1), let us consider the following general class of gauge choices for the vector potential,

$$A_1(x_1, x_2) = -\tfrac{1}{2}B(x_2 - \overline{x}_2) + \partial_1\chi(x_1, x_2), \qquad A_2(x_1, x_2) = \tfrac{1}{2}B(x_1 - \overline{x}_1) + \partial_2\chi(x_1, x_2). \tag{11}$$

Here, $(\overline{x}_1, \overline{x}_2)$ are two constant parameters representing the position of a particular point in the plane, about which configuration space wave functions representing the Fock states to

[5] This is detailed in section 2.

be identified hereafter are centred and localized. Furthermore, $\chi(x_1, x_2)$ is an arbitrary real function representing a possible gauge redefinition of the chosen vector potential. Note that the parameters $(\overline{x}_1, \overline{x}_2)$ could also be absorbed into that gauge transformation function, but it is useful to keep these two constants explicit. Clearly the previous symmetric gauge corresponds to the values $(\overline{x}_1, \overline{x}_2) = (0, 0)$ and $\chi = 0$, while the Landau gauge to $(\overline{x}_1, \overline{x}_2) = (0, 0)$ and $\chi = -Bx_1x_2/2$.

Incidentally, it may easily be checked that the Euler–Lagrange equations of motion that derive from (1) are gauge invariant, namely independent of both $(\overline{x}_1, \overline{x}_2)$ and $\chi(x_1, x_2)$, as should be of course.

For what concerns the classical Hamiltonian formulation of the system, the Hamiltonian reads

$$H = \frac{1}{2m}\left(p_1 + \frac{1}{2}B(x_2 - \overline{x}_2) - \partial_1\chi\right)^2 + \frac{1}{2m}\left(p_2 - \frac{1}{2}B(x_1 - \overline{x}_1) - \partial_2\chi\right)^2, \tag{12}$$

where the phase space variables (x_i, p_i) possess canonical Poisson brackets, $\{x_i, p_j\} = \delta_{ij}$ $(i, j = 1, 2)$. Introducing now the following new parametrization of phase space,

$$u_i = x_i - \overline{x}_i, \qquad \pi_i = p_i - \partial_i\chi(x_i), \tag{13}$$

which defines yet again canonically conjugate pairs of variables,

$$\{u_i, u_j\} = 0, \qquad \{u_i, \pi_j\} = \delta_{ij}, \qquad \{\pi_i, \pi_j\} = 0, \tag{14}$$

one has

$$\begin{aligned} H &= \frac{1}{2m}\left(\pi_1 + \frac{1}{2}Bu_2\right)^2 + \frac{1}{2m}\left(\pi_2 - \frac{1}{2}Bu_1\right)^2 \\ &= \frac{1}{2m}\left(\pi_1^2 + \pi_2^2\right) + \frac{1}{2}m\frac{\omega_c^2}{4}\left(u_1^2 + u_2^2\right) - \frac{1}{2}\omega_c\left(u_1\pi_2 - u_2\pi_1\right). \end{aligned} \tag{15}$$

The system has thereby been brought into the form it has in the symmetric gauge centred at $(x_1, x_2) = (0, 0)$, independently of the original choice of gauge. Note well that this includes the Landau gauge, however now with a choice of phase space canonical coordinates which differs from that which led to (4). This point is addressed more specifically hereafter.

The resolution of the quantized system is now straightforward. Given the quantum commutation relations,

$$[\hat{u}_i, \hat{\pi}_j] = \mathrm{i}\hbar\delta_{ij}\,\mathbb{I}, \qquad \hat{u}_i^\dagger = \hat{u}_i, \qquad \hat{\pi}_i^\dagger = \hat{\pi}_i, \tag{16}$$

one first introduces the cartesian Fock algebra generators,

$$a_i = \frac{1}{2}\sqrt{\frac{B}{\hbar}}\left(\hat{u}_i + \frac{2\mathrm{i}}{B}\hat{\pi}_i\right), \qquad a_i^\dagger = \frac{1}{2}\sqrt{\frac{B}{\hbar}}\left(\hat{u}_i - \frac{2\mathrm{i}}{B}\hat{\pi}_i\right), \tag{17}$$

such that

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}\,\mathbb{I}, \tag{18}$$

while

$$\hat{H} = \tfrac{1}{2}\hbar\omega_c\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right) + \tfrac{1}{2}\mathrm{i}\hbar\omega_c\left(a_1^\dagger a_2 - a_2^\dagger a_1\right). \tag{19}$$

Next one introduces the chiral Fock algebra generators,

$$a_\pm = \tfrac{1}{\sqrt{2}}(a_1 \mp \mathrm{i}a_2), \qquad a_\pm^\dagger = \tfrac{1}{\sqrt{2}}\left(a_1^\dagger \pm \mathrm{i}a_2^\dagger\right), \tag{20}$$

such that

$$\left[a_\pm, a_\pm^\dagger\right] = \mathbb{I}, \qquad \left[a_\pm, a_\mp^\dagger\right] = 0. \tag{21}$$

A direct substitution[6] then finds

$$\hat{H} = \hbar\omega_c\left(a_-^\dagger a_- + \tfrac{1}{2}\right). \tag{22}$$

Consequently, given the orthonormalized Fock state basis $|n_-, n_+\rangle$ ($n_\pm = 0, 1, 2, \ldots$) defined by

$$|n_-, n_+\rangle = \frac{1}{\sqrt{n_-!n_+!}}\left(a_-^\dagger\right)^{n_-}\left(a_+^\dagger\right)^{n_+}|0\rangle, \qquad a_\pm|0\rangle = 0, \qquad \langle 0|0\rangle = 1, \tag{23}$$

these states diagonalize the energy eigenspectrum of the system,

$$\hat{H}|n_-, n_+\rangle = E(n_-)\,|n_-, n_+\rangle, \qquad E(n_-) = \hbar\omega_c\left(n_- + \tfrac{1}{2}\right). \tag{24}$$

Hence indeed the same energy spectrum as in the Landau gauge is obtained, however now with a countable basis of eigenstates which are all normalizable and localized in the plane. More specifically, it may be shown [8] that in the configuration space representation the wave functions of these chiral Fock states are given as

$$\langle x_1, x_2|n_-, n_+\rangle = \frac{(-1)^m}{\sqrt{2\pi\hbar}}\sqrt{\frac{m!}{(m+|\ell|)!}}\,u^{|\ell|/2}\,\mathrm{e}^{\mathrm{i}\ell\theta}\,\mathrm{e}^{-\frac{1}{2}u}\,L_m^{|\ell|}(u), \tag{25}$$

where $\ell = n_+ - n_-$, $m = \min(n_-, n_+) = n_- + (\ell - |\ell|)/2$ and $L_m^{|\ell|}(u)$ are the generalized Laguerre polynomials, while

$$u = \frac{m\omega_c}{2\hbar}[(x_1 - \overline{x}_1)^2 + (x_2 - \overline{x}_2)^2], \qquad \mathrm{e}^{\mathrm{i}\theta} = \frac{(x_1 - \overline{x}_1) + \mathrm{i}(x_2 - \overline{x}_2)}{\sqrt{(x_1 - \overline{x}_1)^2 + (x_2 - \overline{x}_2)^2}}. \tag{26}$$

Clearly all these states are thus indeed localized and centred at the point $(\overline{x}_1, \overline{x}_2)$, and normalizable, independently of the chosen gauge for the vector potential, including the Landau gauge. This result is achieved by having identified the appropriate canonical phase space transformation which undoes any gauge transformation away from the symmetric gauge, while at the same time moving the set of localized Fock states to be centred at any given point in the plane.

2.2. *The solution in the Landau gauge*

In terms of the general parametrization for a gauge choice in (11), the Landau gauge as defined in the introduction corresponds to the function

$$\chi(x_1, x_2) = -\tfrac{1}{2}B(x_1 - \overline{x}_1)(x_2 + \overline{x}_2). \tag{27}$$

Consequently, one then finds

$$\begin{aligned} \pi_1 &= p_1 + \tfrac{1}{2}B(x_2 + \overline{x}_2), \qquad \pi_1 + \tfrac{1}{2}Bu_2 = \pi_1 + \tfrac{1}{2}B(x_2 - \overline{x}_2) = p_1 + Bx_2, \\ \pi_2 &= p_2 + \tfrac{1}{2}B(x_1 - \overline{x}_1), \qquad \pi_2 - \tfrac{1}{2}Bu_1 = \pi_2 - \tfrac{1}{2}B(x_1 - \overline{x}_1) = p_2, \end{aligned} \tag{28}$$

so that indeed,

$$H = \frac{1}{2m}(p_1 + Bx_2)^2 + \frac{1}{2m}p_2^2. \tag{29}$$

Given these relations in the Landau gauge, it is now possible to express the operators $\hat{x}_i$ and $\hat{p}_i$ in terms of the cartesian and chiral Fock operators introduced above. One then finds

$$\hat{x}_1 = \overline{x}_1 + \sqrt{\frac{\hbar}{B}}\left(a_1 + a_1^\dagger\right), \qquad \hat{x}_2 = \overline{x}_2 + \sqrt{\frac{\hbar}{B}}\left(a_2 + a_2^\dagger\right), \tag{30}$$

[6] The inverse relations expressing $\hat{u}_i$ and $\hat{\pi}_i$ in terms of $(a_\pm, a_\pm^\dagger)$ are easily worked out.

$$\hat{p}_1 = -\frac{1}{2}\mathrm{i}\sqrt{\hbar B}\big(a_1 - a_1^\dagger\big) - \frac{1}{2}\sqrt{\hbar B}\big(a_2 + a_2^\dagger\big) - B\overline{x}_2,$$
$$\hat{p}_2 = -\frac{1}{2}\mathrm{i}\sqrt{\hbar B}\big(a_2 - a_2^\dagger\big) - \frac{1}{2}\sqrt{\hbar B}\big(a_1 + a_1^\dagger\big). \qquad (31)$$

We then have

$$\hat{p}_2 = -\sqrt{\frac{\hbar B}{2}}\big(a_- + a_-^\dagger\big), \qquad \hat{x}_2 + \frac{1}{B}\hat{p}_1 = -\mathrm{i}\sqrt{\frac{\hbar}{2B}}\big(a_- - a_-^\dagger\big), \qquad (32)$$

so that the $(a, a^\dagger)$ Fock generators defined in (5) in the Landau gauge correspond to

$$a = -\mathrm{i}a_-, \qquad a^\dagger = \mathrm{i}a_-^\dagger. \qquad (33)$$

Hence, we have indeed that

$$\hat{H} = \hbar\omega_c\big(a_-^\dagger a_- + \tfrac{1}{2}\big) = \hbar\omega_c\left(a^\dagger a + \tfrac{1}{2}\right), \qquad (34)$$

explaining why the same values for the energy spectrum are obtained in both constructions for the quantum solution. However, in the discussion as recalled in the introduction, the degeneracy of Landau levels is accounted for in terms of the eigenstates of $\hat{p}_1$, namely,

$$\hat{p}_1 = -\mathrm{i}\sqrt{\frac{\hbar B}{2}}\left(a_+ - a_+^\dagger\right) - B\overline{x}_2, \qquad (35)$$

rather than the Fock states $|n_+\rangle$ of the Fock algebra $\left(a_+, a_+^\dagger\right)$ as obtained in the previous general solution irrespective of the choice of gauge. Therefore, when expressed in terms of these Fock states, the solution to the eigenvalue equation,

$$\hat{p}_1|p_1\rangle = p_1|p_1\rangle, \qquad p_1 \in \mathbb{R}, \qquad (36)$$

involves an infinite linear combination of all Fock states $|n_+\rangle$ which is not normalizable.

In other words, the reason why the usual solution to the Landau problem in the Landau gauge leads to states which, within each of the Landau levels, are not normalizable nor localized, is not at all related to that particular choice of gauge. Rather, it is because that choice of gauge naturally leads one to use such a canonical parametrization of phase space which upon its canonical quantization produces a basis of energy eigenstates which, in each Landau level, are not normalizable nor localized. However, this singular character in the choice of basis within each Landau level is avoided by an appropriate canonical transformation which upon its canonical quantization produces a basis of energy eigenstates which, as Fock states, are all normalizable and localized irrespective of the choice of gauge. It is thus coincidental that precisely in the Landau gauge, the generic canonical phase space parametrization valid for any choice of gauge is just not manifest enough, so that one is led rather onto a path towards another construction of a solution for energy eigenstates which are no longer normalizable nor localized.

This analysis thus also shows that it is preferable to consider in all cases a parametrization of the general choice of gauge as in (11), which in effect is a gauge transformed form of the symmetric gauge. One is then assured that if energy eigenstates are not normalizable or localized, there is actually a physical justification or meaning to such a singular character, rather than being due to some inappropriate choice of canonical parametrization of phase space.

3. The Landau problem with a quadratic and linear potential

Given the above understanding of the preferred choice of gauge, let us now consider an extension of the Landau problem which includes an interaction potential energy, $V(x_1, x_2)$, still leading to linear equations of motion, whether at the classical level or the quantum one in the Heisenberg picture. In order to remain consistent with the rotational invariance of the original problem, this potential consists of a spherically symmetric harmonic well of angular frequency $\omega_0 > 0$ centred at the origin $(x_1, x_2) = (0, 0)$, to which a linear term is added, lying—by an appropriate choice of planar coordinates (x_1, x_2)—in the x_2 direction,

$$V(x_1, x_2) = \frac{1}{2} m\omega_0^2 \left(x_1^2 + x_2^2\right) + \gamma x_2. \tag{37}$$

Here γ is a real constant parameter, setting the strength of a constant pull onto the particle in the $(-x_2)$ direction (for positive γ). This linear potential may correspond, for instance, to a constant electric field lying inside the plane and along the x_2 direction. Another possibility is a gravitational potential term if the plane is tilted with respect to the horizontal direction by some angle α, in which case one has $\gamma = mg \cos\alpha$ if x_2 increases upwards, $g > 0$ being the gravitational acceleration. These two examples thus indicate to which types of physical configurations such a linear potential could correspond.

Choosing for the vector potential the symmetric gauge as in (8) does not lead to a straightforward resolution of either the Hamiltonian or the quantum dynamics. Indeed, since that choice is centred onto the point $(\overline{x}_1, \overline{x}_2) = (0, 0)$, it clashes with the fact that because of the linear term in the potential energy, the total potential energy—still spherically symmetric—is centred onto a minimal position given by

$$\overline{x}_1 = 0, \qquad \overline{x}_2 = -\frac{\gamma}{m\omega_0^2}, \tag{38}$$

since

$$V(x_1, x_2) = \frac{1}{2} m\omega_0^2 \left(x_1^2 + \left(x_2 + \frac{\gamma}{m\omega_0^2}\right)^2\right) - \frac{\gamma^2}{2m\omega_0^2}. \tag{39}$$

Obviously classical trajectories will then be centred onto that static average position in the plane. Consequently, it is preferable to adapt the choice of symmetric gauge in the following way

$$A_1(x_1, x_2) = -\frac{1}{2} B \left(x_2 + \frac{\gamma}{m\omega_0^2}\right), \qquad A_2(x_1, x_2) = +\frac{1}{2} B x_1. \tag{40}$$

The Hamiltonian is then of the form,

$$\begin{aligned} H &= \frac{1}{2m}\left(p_1 + \frac{1}{2}B\left(x_2 + \frac{\gamma}{m\omega_0^2}\right)\right)^2 + \frac{1}{2m}\left(p_2 - \frac{1}{2}Bx_1\right)^2 + \frac{1}{2}m\omega_0^2\left(x_1^2 + x_2^2\right) + \gamma x_2 \\ &= \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{1}{2}m\omega^2\left(x_1^2 + \left(x_2 + \frac{\gamma}{m\omega_0^2}\right)^2\right) \\ &\quad - \frac{1}{2}\omega_c\left(x_1 p_2 - \left(x_2 + \frac{\gamma}{m\omega_0^2}\right)p_1\right) - \frac{\gamma^2}{2m\omega_0^2}, \end{aligned} \tag{41}$$

where $\omega = \sqrt{\omega_0^2 + \omega_c^2/4}$.

The diagonalization of this quantum Hamiltonian is now straightforward enough.[7] Given the Heisenberg algebra $[\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}\mathbb{I}$, let us first introduce the following cartesian Fock

[7] Had one not chosen the symmetric gauge centred on the point in (38), there would have remained terms linear in $\hat{p}_1$ in $\hat{H}$ spoiling the simplicity of the present solution.

algebra, this time in terms of the effective angular frequency ω rather than the cyclotron one, ω_c,

$$a_1 = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_1 + \frac{\mathrm{i}}{m\omega}\hat{p}_1\right), \qquad a_1^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_1 - \frac{\mathrm{i}}{m\omega}\hat{p}_1\right),$$
$$a_2 = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_2 + \frac{\gamma}{m\omega_0^2}\mathbb{I} + \frac{\mathrm{i}}{m\omega}\hat{p}_2\right), \qquad a_2^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_2 + \frac{\gamma}{m\omega_0^2}\mathbb{I} - \frac{\mathrm{i}}{m\omega}\hat{p}_2\right), \tag{42}$$

which are such that,

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}\,\mathbb{I}. \tag{43}$$

Introduce now once again the chiral or helicity Fock algebra operators

$$a_\pm = \frac{1}{\sqrt{2}}\left(a_1 \mp \mathrm{i}a_2\right), \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}}\left(a_1^\dagger \pm \mathrm{i}a_2^\dagger\right), \tag{44}$$

$$\left[a_\pm, a_\pm^\dagger\right] = \mathbb{I}. \tag{45}$$

A simple substitution then finds for the quantum Hamiltonian,

$$\hat{H} = \hbar\omega\left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right) - \frac{1}{2}\hbar\omega_c\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) - \frac{\gamma^2}{2m\omega_0^2}, \tag{46}$$

which is thus diagonalized on the basis of Fock states $|n_-, n_+\rangle$ ($n_\pm = 0, 1, 2, \ldots$) constructed out of the chiral Fock algebra,

$$\hat{H}|n_-, n_+\rangle = E(n_-, n_+)\,|n_-, n_+\rangle, \qquad E(n_-, n_+) = \hbar\omega_+ n_- + \hbar\omega_- n_+ + \hbar\omega - \frac{\gamma^2}{2m\omega_0^2}, \tag{47}$$

where

$$\omega_+ = \omega + \tfrac{1}{2}\omega_c, \qquad \omega_- = \omega - \tfrac{1}{2}\omega_c. \tag{48}$$

The energy eigenspectrum thus consists of normalizable and localized states. As a matter of fact, the configuration space wave functions, $\langle x_1, x_2|n_-, n_+\rangle$, of these chiral Fock states are given as in (25), with this time the following definition for the two variables u and θ,

$$u = \frac{m\omega}{\hbar}\left[x_1^2 + \left(x_2 + \frac{\gamma}{m\omega_0^2}\right)^2\right], \qquad \mathrm{e}^{\mathrm{i}\theta} = \frac{x_1 + \mathrm{i}\left(x_2 + \frac{\gamma}{m\omega_0^2}\right)}{\sqrt{x_1^2 + \left(x_2 + \frac{\gamma}{m\omega_0^2}\right)^2}}. \tag{49}$$

It is also of interest to consider the time evolution of the quantum phase space operators $(\hat{x}_i, \hat{p}_i)$ in the Heisenberg picture. Given the above expression for the quantum Hamiltonian, the time evolution of each of the Fock algebra operators is readily identified, leading to

$$\hat{x}_1(t) = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\left(a_+\,\mathrm{e}^{-\mathrm{i}\omega_- t} + a_-\,\mathrm{e}^{-\mathrm{i}\omega_+ t} + a_+^\dagger\,\mathrm{e}^{\mathrm{i}\omega_- t} + a_-^\dagger\,\mathrm{e}^{\mathrm{i}\omega_+ t}\right),$$
$$\hat{x}_2(t) = -\frac{\gamma}{m\omega_0^2}\mathbb{I} + \frac{1}{2}\mathrm{i}\sqrt{\frac{\hbar}{m\omega}}\left(a_+\,\mathrm{e}^{-\mathrm{i}\omega_- t} - a_-\,\mathrm{e}^{-\mathrm{i}\omega_+ t} - a_+^\dagger\,\mathrm{e}^{\mathrm{i}\omega_- t} + a_-^\dagger\,\mathrm{e}^{\mathrm{i}\omega_+ t}\right),$$
$$\hat{p}_1(t) = -\frac{1}{2}\mathrm{i}m\omega\sqrt{\frac{\hbar}{m\omega}}\left(a_+\,\mathrm{e}^{-\mathrm{i}\omega_- t} + a_-\,\mathrm{e}^{-\mathrm{i}\omega_+ t} - a_+^\dagger\,\mathrm{e}^{\mathrm{i}\omega_- t} - a_-^\dagger\,\mathrm{e}^{\mathrm{i}\omega_+ t}\right),$$
$$\hat{p}_2(t) = \frac{1}{2}m\omega\sqrt{\frac{\hbar}{m\omega}}\left(a_+\,\mathrm{e}^{-\mathrm{i}\omega_- t} - a_-\,\mathrm{e}^{-\mathrm{i}\omega_+ t} + a_+^\dagger\,\mathrm{e}^{\mathrm{i}\omega_- t} - a_-^\dagger\,\mathrm{e}^{\mathrm{i}\omega_+ t}\right). \tag{50}$$

Of course, these expressions provide the explicit solutions to the linear Hamiltonian equations of motion of the system, whether at the classical level or the quantum level in the Heisenberg picture. Note well that all the above operators $\left(a_\pm, a_\pm^\dagger\right)$ are defined by the initial Heisenberg commutation relations specified either in the Schrödinger picture, or the Heisenberg picture at initial time $t = 0$.

All these expressions reproduce also those of the ordinary Landau problem of section 2.1, provided however the limits in $\omega_0^2 \to 0$ and $\gamma \to 0$ are taken appropriately. First, the linear potential term needs to be removed, $\gamma \to 0$, and only then is the spherically symmetric well to be flattened out, $\omega_0^2 \to 0$. All the expressions above are then smoothly mapped back to those of section 2.1. In other words, by first bringing the equilibrium point of the total spherically symmetric potential back to the origin of the plane, $(x_1, x_2) = (0, 0)$, namely by first removing the linear contribution, and only then removing the spherical well, one reproduces the original Landau problem.

However when the limits are considered in the reverse order, one immediately runs into singularities. Indeed, note that when first the spherical well is flattened out while still keeping the constant force acting on the particle, $\omega_0^2 \to 0$ but $\gamma \neq 0$, singularities in the quantity $\gamma\big/\left(m\omega_0^2\right)$ arise and the operators $\left(a_2, a_2^\dagger\right)$, and hence $\left(a_\pm, a_\pm^\dagger\right)$, are then no longer well defined, nor is thus the general solution in (50) and the chiral Fock states $|n_-, n_+\rangle$.

Classically, by first removing the spherical well the particle is being set free—it is no longer confined within the well—and being subjected to a constant force inside the plane in conjunction with the magnetic force which is always perpendicular to its velocity, the net result is a circular motion around a magnetic centre which rather than being static as in the ordinary Landau problem now moves at a constant velocity in a direction perpendicular to both the magnetic field and the constant force, namely in our case along the x_1 direction with the velocity,

$$\dot{x}_1^c = -\frac{\gamma}{B}, \qquad \dot{x}_2^c = 0. \tag{51}$$

In terms of the above solution considered at the classical level, in the limit $\omega_0^2 \to 0$ one has $\omega_- \to 0$, so that the actual classical solution acquires a linear time dependence—the one describing the motion of its magnetic centre at a constant velocity—combined with a periodic circular motion of angular frequency $\omega_+ \to \omega_c$ once again, about that moving magnetic centre. Applying a Galilei boost taking the system to the inertial frame of the magnetic centre, one recovers the ordinary Landau problem. Note that the motion of the magnetic centre is along the x_1 axis, but with a value for x_2 which is a function of the initial conditions for the classical trajectory.

Rather than considering trying to apply to the above quantum solution, in particular the construction of its chiral Fock states, a singular limiting procedure which at the classical level produces out of the general solutions in (50) the correct ones when $\omega_0^2 = 0$, since the Hamiltonian equations of motion are linear and thus identical whether for the classical or the quantum system it is more straightforward to immediately consider the situation with $\omega_0^2 = 0$ and $\gamma \neq 0$ at the classical level, and out of its solutions construct the appropriate realization of the quantum system in that singular case of the extended Landau problem. Such an approach is also simpler than trying to apply to the ordinary Landau problem a Galilei boost from the magnetic centre frame back to the initial frame in which the dynamics of the system is being considered.

4. The Landau problem with a linear potential

In the symmetric gauge, the Lagrangian of the system now reads

$$L = \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \tfrac{1}{2}B\left(\dot{x}_1 x_2 - x_1\dot{x}_2\right) - \gamma x_2, \tag{52}$$

hence the Hamiltonian

$$H = \frac{1}{2m}\left(p_1 + \frac{1}{2}Bx_2\right)^2 + \frac{1}{2m}\left(p_2 - \frac{1}{2}Bx_1\right)^2 + \gamma x_2. \tag{53}$$

Trying to apply to the quantized version of the system the same types of operator redefinitions as those of section 3 runs into the difficulty that the term linear in γx_2 remains non-diagonal in whatever Fock state basis being considered. In order to tackle this issue, in the same way as was discussed in section 2.2 for the ordinary Landau problem in the Landau gauge, first a canonical transformation of phase space parametrization is required, which readily provides at the quantum level the diagonalized Hamiltonian, hence the solution to the quantum dynamics of the system.

Rather than specifying this canonical transformation still at the classical level and in terms of Poisson brackets, let us already define it at the quantum level for the quantum operators and their commutation relations in the Schrödinger picture, or the Heisenberg picture at time $t = 0$. Consider then the following definitions, from the variables $(\hat{x}_i, \hat{p}_i)$ to the variables $\left(\hat{x}_1^c, \hat{x}_2^c, a_-, a_-^\dagger\right)$,

$$\begin{aligned}
\hat{x}_1^c &= \frac{1}{2}\hat{x}_1 + \frac{1}{m\omega_c}\hat{p}_2, \\
\hat{x}_2^c &= \frac{1}{2}\hat{x}_2 - \frac{1}{m\omega_c}\hat{p}_1 - \frac{\gamma}{m\omega_c^2}, \\
a_- &= \sqrt{\frac{m\omega_c}{2\hbar}}\left(\frac{1}{2}\hat{x}_1 - \frac{1}{m\omega_c}\hat{p}_2\right) + \frac{\mathrm{i}}{m\omega_c}\sqrt{\frac{m\omega_c}{2\hbar}}\left(\hat{p}_1 + \frac{1}{2}m\omega_c\hat{x}_2 + \frac{\gamma}{\omega_c}\,\mathbb{I}\right), \\
a_-^\dagger &= \sqrt{\frac{m\omega_c}{2\hbar}}\left(\frac{1}{2}\hat{x}_1 - \frac{1}{m\omega_c}\hat{p}_2\right) - \frac{\mathrm{i}}{m\omega_c}\sqrt{\frac{m\omega_c}{2\hbar}}\left(\hat{p}_1 + \frac{1}{2}m\omega_c\hat{x}_2 + \frac{\gamma}{\omega_c}\,\mathbb{I}\right).
\end{aligned} \tag{54}$$

The inverse relations are

$$\begin{aligned}
\hat{x}_1 &= \hat{x}_1^c + \sqrt{\frac{\hbar}{2m\omega_c}}\left(a_- + a_-^\dagger\right), \\
\hat{x}_2 &= \hat{x}_2^c - \mathrm{i}\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_- - a_-^\dagger\right), \\
\hat{p}_1 &= -\frac{\gamma}{\omega_c}\mathbb{I} - \frac{1}{2}m\omega_c\hat{x}_2^c - \frac{1}{2}\mathrm{i}m\omega_c\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_- - a_-^\dagger\right), \\
\hat{p}_2 &= \frac{1}{2}m\omega_c\hat{x}_1^c - \frac{1}{2}m\omega_c\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_- + a_-^\dagger\right).
\end{aligned} \tag{55}$$

It then follows that the two sectors $\left(\hat{x}_1^c, \hat{x}_2^c\right)$ and $\left(a_-, a_-^\dagger\right)$ commute with one another, while we have

$$\left[\hat{x}_1^c, \hat{x}_2^c\right] = -\frac{\mathrm{i}\hbar}{B}\,\mathbb{I}, \qquad \left[a_-, a_-^\dagger\right] = \mathbb{I}. \tag{56}$$

Hence indeed this reparametrization of phase space is a canonical transformation preserving canonical Poisson brackets (one may rescale, say $\hat{x}_2^c$, by the factor $(-B = -m\omega_c)$, if one prefers). Related to the magnetic centre sector, $\left(\hat{x}_1^c, \hat{x}_2^c\right)$, we also have the following Fock algebra generators,

$$\hat{x}_1^c = \sqrt{\frac{\hbar}{2m\omega_c}}\left(a_+ + a_+^\dagger\right), \qquad \hat{x}_2^c = \mathrm{i}\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_+ - a_+^\dagger\right), \tag{57}$$

which are such that

$$\left[a_+, a_+^\dagger\right] = \mathbb{I}. \tag{58}$$

These operators $\left(a_+, a_+^\dagger\right)$ correspond to the right-handed chiral mode of the ordinary Landau problem which in that context has no time dependence, but acquires one in the present case because of the constant force of strength γ which indeed sets into motion the magnetic centre. Note that $\hat{x}_1^c$ corresponds to the magnetic centre position along the x_1 axis, while the contribution $\hat{x}_2^c$ to $\hat{x}_2$ corresponds to its position along the x_2 axis. The contribution $(-\gamma/\omega_c = -m\gamma/B)$ to $\hat{p}_1$ corresponds to the velocity momentum of the magnetic centre, $m\dot{x}_1^c$. Finally, $\left(a_-, a_-^\dagger\right)$ correspond to the left-handed chiral rotating mode with angular frequency ω_c of the ordinary Landau problem in the magnetic centre inertial frame

A direct substitution of these relations in the Hamiltonian finds

$$\hat{H} = \hbar\omega_c\left(a_-^\dagger a_- + \frac{1}{2}\right) + \gamma\hat{x}_2^c + \frac{1}{2}m\left(\frac{\gamma}{B}\right)^2. \tag{59}$$

The physical meaning of each of these contributions should be clear enough. The very last term corresponds to the kinetic energy of the magnetic centre moving at constant velocity of norm $|\gamma|/B$. The term before the last represents the potential energy along the x_2 direction, γx_2, of the magnetic centre position along that axis. And finally the very first contribution with the two terms in parentheses measures the excitation energy of the usual Landau levels of the ordinary Landau problem, as seen from the magnetic centre inertial frame.

This expression for the quantum Hamiltonian also makes it clear which basis of states diagonalizes that operator, hence solves the quantum dynamics of the system. Given the Fock states $|n_-\rangle$ associated with the $\left(a_-, a_-^\dagger\right)$ Fock algebra, and position eigenstates, $\hat{x}_2^c\left|x_2^c\right\rangle = x_2^c\left|x_2^c\right\rangle$, for the magnetic centre position along the x_2 axis, the basis of the space of quantum states which diagonalizes the dynamics is spanned by the states $\left|n_-, x_2^c\right\rangle$ $(n_- = 0, 1, 2, \ldots, x_2^c \in \mathbb{R})$, with the normalization,

$$\left\langle n_-, x_2^c \middle| m_-, x'^c_2\right\rangle = \delta_{n_-,m_-}\,\delta\left(x_2^c - x'^c_2\right). \tag{60}$$

One has

$$\hat{H}\left|n, x_2^c\right\rangle = E\left(n, x_2^c\right)\left|n, x_2^c\right\rangle, \qquad E\left(n, x_2^c\right) = \hbar\omega_c\left(n + \frac{1}{2}\right) + \gamma x_2^c + \frac{1}{2}m\left(\frac{\gamma}{B}\right)^2. \tag{61}$$

The fact that the magnetic centre component of these quantum states is not normalizable makes now perfect physical sense. Indeed, the motion of that magnetic centre is that of a free particle with a predetermined velocity set by the ratio $(-\gamma/B)$ in the x_1 direction. Hence in the configuration space representation states possess wave functions with a plane wave component in that direction, which is not normalizable and leads to the above δ function normalization for energy eigenstates. Having chosen from the outset not to work in the Landau gauge guarantees without ambiguity that this lack of normalizability is indeed related to a physical feature of the solution rather than a not totally appropriate choice of phase space parametrization.

Given the Hamiltonian, it is also possible to determine the time dependence of the phase space operators in the Heisenberg picture. One finds

$$\hat{x}_1(t) = \hat{x}_1^c - \frac{\gamma}{B} t\,\mathbb{I} + \sqrt{\frac{\hbar}{2m\omega_c}}\left(a_-\,\mathrm{e}^{-\mathrm{i}\omega_c t} + a_-^\dagger\,\mathrm{e}^{\mathrm{i}\omega_c t}\right),$$

$$\hat{x}_2(t) = \hat{x}_2^c - \mathrm{i}\sqrt{\frac{\hbar}{2m\omega_c}}\left(a_-\,\mathrm{e}^{-\mathrm{i}\omega_c t} - a_-^\dagger\,\mathrm{e}^{\mathrm{i}\omega_c t}\right),$$

$$\hat{p}_1(t) = -\frac{\gamma}{\omega_c}\,\mathbb{I} - \frac{1}{2}m\omega_c\,\hat{x}_2^c - \frac{1}{2}\mathrm{i}m\omega_c\sqrt{\frac{\hbar}{m\omega_c}}\left(a_-\,\mathrm{e}^{-\mathrm{i}\omega_c t} - a_-^\dagger\,\mathrm{e}^{\mathrm{i}\omega_c t}\right),$$

$$\hat{p}_2(t) = \frac{1}{2}m\omega_c\,\hat{x}_1^c - \frac{1}{2}\gamma t\,\mathbb{I} - \frac{1}{2}m\omega_c\sqrt{\frac{\hbar}{m\omega_c}}\left(a_-\,\mathrm{e}^{-\mathrm{i}\omega_c t} - +a_-^\dagger\,\mathrm{e}^{\mathrm{i}\omega_c t}\right), \qquad (62)$$

namely,

$$\hat{x}_1^c(t) = \hat{x}_1^c - \frac{\gamma}{B} t\,\mathbb{I}, \qquad \hat{x}_2^c(t) = \hat{x}_2^c. \qquad (63)$$

In the expressions for $\hat{x}_i(t)$, one may recognize the solution to the ordinary Landau problem (the terms involving a_- and $a_-^\dagger$), valid in the magnetic centre inertial frame, to which is added the Galilei boost with the constant velocity of the magnetic centre towards the inertial frame with the potential energy γx_2, and the initial position of that magnetic centre along both the x_1 and x_2 axes. Incidentally, and as was indicated at the end of the previous section, this is in fact how the change of variables (54) was identified initially. Writing out the classical solution for $x_i(t)$ precisely in that way, and then identifying the ensuing expressions for $p_i(t)$ given that

$$p_1(t) = m\dot{x}_1(t) + \tfrac{1}{2}Bx_2(t), \qquad p_2(t) = m\dot{x}_2(t) - \tfrac{1}{2}Bx_1(t), \qquad (64)$$

all in a manner that meets all Hamiltonian equations of motion, the appropriate operators in the Heisenberg picture are identified. Upon substitution into the quantum Hamiltonian, one then is bound to find the quantum solution for it as well.

Note also that in the same spirit as that of the entire discussion so far, the above solution of the singular Landau problem extended with a linear potential has remained purely algebraic, without the need to solve the differential Schrödinger wave equation. To the best of our knowledge, this specific approach and construction for the extended singular Landau problem is not available in the literature.

As a passing remark of some interest as well, note that given a projection onto any subspace of Hilbert space corresponding to a Landau sector at a fixed level n_-, namely onto the subspace spanned by the states $\left|n_-, x_2^c\right\rangle$ for a fixed n_- and for all $x_2^c \in \mathbb{R}$, in effect the only remaining degrees of freedom are those of the magnetic centre, $\hat{x}_1^c$ and $\hat{x}_2^c$, which obey the commutation relation of the ordinary Moyal–Voros plane of noncommutative geometry [5, 6], $\left[\hat{x}_1^c, \hat{x}_2^c\right] = -\mathrm{i}(\hbar/B)\mathbb{I}$. This result is readily established through the present discussion without the need of any actual calculation of projected matrix elements, in contradistinction to the usual derivation of this result available in the literature [9].

Finally, it now becomes feasible without much difficulty to explicitly work out the configuration space wave functions for all energy eigenstates, given the expressions of the operators $\left(\hat{x}_1^c, \hat{x}_2^c, a_-, a_-^\dagger\right)$ in terms of $(\hat{x}_i, \hat{p}_i)$. This task would be a great deal more involved were one to consider from the outset the differential Schrödinger equation eigenvalue problem given the Hamiltonian in the form of (53). Without going here into the details of the calculation, let us only mention that in a first step one works out the wave function for the lowest Landau sector, $\left|n_- = 0, x_2^c\right\rangle$, as a function of x_2^c. Applying then the operator $a_-^\dagger$ on that solution, one readily finds the wave functions for all states $\left|n_-, x_2^c\right\rangle$, including their normalization. When normalizing the position eigenstates of the configuration space basis as is usual,

$$\langle x_1, x_2 | x_1', x_2' \rangle = \delta(x_1 - x_1')\,\delta(x_2 - x_2'), \qquad \hat{x}_i\,|x_1, x_2\rangle = x_i\,|x_1, x_2\rangle, \tag{65}$$

one finds

$$\langle x_1, x_2 | n_-, x_2^c \rangle = \left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4} \left(\frac{m\omega_c}{2\pi\hbar}\right)^{1/2} \frac{(-\mathrm{i})^n}{\sqrt{2^n\, n!}}$$

$$\times\, \mathrm{e}^{-\mathrm{i}\frac{m\omega_c}{\hbar} x_1 (x_2^c - \frac{1}{2}x_2 + \frac{\gamma}{m\omega_c^2})}\, \mathrm{e}^{-\frac{m\omega_c}{2\hbar}\left(x_2 - x_2^c\right)^2} H_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(x_2 - x_2^c\right)\right), \tag{66}$$

$H_n(u)$ being the Hermite polynomial of order n.

Hence these energy eigenstates are localized only in the x_2 direction, while in the x_1 direction they are totally delocalized and non-normalizable since they propagate in time in that direction as a free particle of predetermined constant velocity $(-\gamma/B)$. The probability density of these states thus also looks like a series of $(n+1)$ parallel stripes with exponentially smooth edges, and invariant under translations along the x_1 axis.

Having constructed a complete solution of this singular Landau problem extended by a linear potential, note how all these results are smooth in the parameter γ, and indeed reproduce in the limit $\gamma \to 0$ those of the ordinary Landau problem. Clearly in that limit, the states $|n_-, x_2^c\rangle$ remain nonnormalizable, because the right-handed chiral sector $\left(a_+, a_+^\dagger\right)$ has been diagonalized rather in terms of the operator $\hat{x}_2^c$, namely by having required the magnetic centre position to be sharp in the x_2 direction, hence totally delocalizing its position along x_1, since the two coordinates of the magnetic centre obey a Heisenberg algebra and do not commute. However in the limit $\gamma = 0$ each of the Landau sectors distinguished by n_- becomes once again energy degenerate, allowing for another choice of energy eigenstate basis in the magnetic centre sector. Choosing for it once again the orthonormalized Fock state basis $|n_-, n_+\rangle$, one recovers precisely exactly the same solution as that constructed in section 2.1 (with $(\overline{x}_1, \overline{x}_2) = (0, 0)$). However the construction of the present section is useful even for the ordinary Landau problem, when a sharp rectilinear edge is introduced in the plane, as we now discuss.

5. The Landau problem in the half-plane with a linear potential

A noteworthy feature of the energy spectrum (61) is that it is unbounded below, and yet the quantum system (as well as the classical system) remains stable because the magnetic force combines with the constant force of strength γ to keep the particle rotating periodically around a magnetic centre that moves at constant velocity along the x_1 axis. Clearly, the reason for this unboundedness in the energy is that the particle may 'fall off the plane' in one of the x_2 directions, so to say, a way of putting this fact which is particularly appropriate in the case that the constant force is indeed that of gravity.

A manner in which to avoid this unboundedness is to restrict the range of x_2, namely consider now the Landau problem on the half-plane still with the linear potential. Assuming now that $\gamma > 0$, let us therefore restrict to the $x_2 \geqslant X_2$ half-plane for some value of $X_2 \in \mathbb{R}$, and reconsider the solution of the quantized system. Such a situation is also of physical interest. Given that the Landau problem is of relevance to the quantum Hall effect, in particular in its integer and even fractional manifestations, combining the magnetic field with a constant force acting inside the plane, be it electrical or gravitational, may allow for interesting properties of that collective quantum fermion phenomenon to manifest themselves. In the gravitational context by tilting the quantum Hall device towards the vertical direction, one is setting up a gravitational quantum well in combination with the quantum Hall effect. Given that the energy quantization of gravitational quantum states in a gravitational well has been observed already with ultra-cold neutrons [10], a quantum Hall set-up may provide an interesting alternative to such measurements, provided that the orders of magnitude for any effect are large enough to be observable. Of course, given the weakness of gravity, an electric field stands a much better chance to display any such interesting effects.

Clearly the change of variable specified in (54) is still in order in this case, leading to the Hamiltonian in (59). However there is a subtlety now, given the boundary at $x_2 = X_2$. Since one has to restrict now to the quantum Hilbert space of configuration space wave functions that vanish at that boundary as well as inside the excluded domain, $x_2 < X_2$, the conjugate momentum operator $\hat{p}_2$ does not possess a self-adjoint extension and is in fact only symmetric on that space [11]. Indeed, as the differential operator $(-i\hbar\partial_2)$, the operator $\hat{p}_2$ maps quantum states out of that Hilbert space, while we have, for any two states $|\psi\rangle$ and $|\varphi\rangle$ represented by functions $\psi(x_2)$ and $\varphi(x_2)$ in the $(\hat{x}_2, \hat{p}_2)$ sector,

$$\begin{aligned}\langle\varphi|\hat{p}_2\psi\rangle &= \int_{X_2}^{+\infty} \mathrm{d}x_2\, \varphi^*(x_2)\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x_2}\psi(x_2)\right)\\ &= -i\hbar\int_{X_2}^{+\infty}\mathrm{d}(\varphi^*(x_2)\psi(x_2)) + \int_{X_2}^{+\infty}\mathrm{d}x_2\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x_2}\varphi(x_2)\right)^*\psi(x_2)\\ &= -i\hbar\int_{X_2}^{+\infty}\mathrm{d}(\varphi^*(x_2)\psi(x_2)) + \langle\hat{p}_2\varphi|\psi\rangle. \end{aligned} \tag{67}$$

Given that both wave functions $\psi(x_2)$ and $\varphi(x_2)$ mush vanish at $x_2 = X_2$ and $x_2 \to +\infty$ (the latter condition applies since states must be normalizable at least in the x_2 direction), it follows that $\hat{p}_2$ is indeed a symmetric operator.

However among those operators contributing to the quantum Hamiltonian still given as in (59), this ambiguity affects only the $\left(a_-, a_-^\dagger\right)$ operators, which are thus no longer adjoints of one another, since each maps outside the considered space of quantum states. However, the other operator involved in diagonalizing the Hamiltonian, namely $\hat{x}_2^c$, is not affected by that lack of self-adjointness in $\hat{p}_2$ since it is given as

$$\hat{x}_2^c = \frac{1}{2}\hat{x}_2 - \frac{1}{m\omega_c}\hat{p}_1 - \frac{\gamma}{m\omega_c^2}, \tag{68}$$

which is an expression that does not involve the operator $\hat{p}_2$, in contradistinction to the operators $\left(a_-, a_-^\dagger\right)$. Consequently, one may still consider the space of eigenstates of $\hat{x}_2^c$, whose wave functions are given as in the construction of the previous section, as a factor in the tensor product structure providing a basis of energy eigenstates. For the remaining separable factor in that tensor product, even though one may no longer exploit the existence of a Fock vacuum annihilated by a_- in order to diagonalize the Hamiltonian, it is rather the latter Hamiltonian which needs diagonalization, a procedure which does not necessarily require a construction of Fock states representing a Fock algebra which in the present case does not exist. As will be seen hereafter, in contradistinction to the $\left(a_-, a_-^\dagger\right)$ operators, the Hamiltonian operator itself is not affected by that issue and does possess a self-adjoint extension [11].

Writing the quantum Hamiltonian in the form

$$\hat{H} = \frac{1}{2}\hbar\omega_c\left(a_-^\dagger a_- + a_- a_-^\dagger\right) + \gamma\hat{x}_2^c + \frac{1}{2}m\left(\frac{\gamma}{B}\right)^2, \tag{69}$$

and using the explicit expressions for a_- and $a_-^\dagger$ in (54), one undoes part of the canonical transformation to find

$$\hat{H} = \frac{1}{2}m\omega_c^2\left(\frac{1}{2}\hat{x}_1 - \frac{1}{m\omega_c}\hat{p}_2\right)^2 + \frac{1}{2m}\left(\hat{p}_1 + \frac{1}{2}m\omega_c\hat{x}_2 + \frac{\gamma}{\omega_c}\right)^2 + \gamma\hat{x}_2^c + \frac{1}{2}m\left(\frac{\gamma}{B}\right)^2. \tag{70}$$

Let us now consider the diagonalization of this operator by working in the configuration space wave function representation for quantum states, $\psi(x_1, x_2)$. Since energy eigenstates are certainly eigenstates of $\hat{x}_2^c$, their wave functions certainly separate as

$$\psi_{E,x_2^c}(x_1, x_2) = \mathrm{e}^{-\mathrm{i}\frac{m\omega_c}{\hbar}x_1(x_2^c - \frac{1}{2}x_2 + \frac{\gamma}{m\omega_c^2})}\,\varphi_{E,x_2^c}(x_2), \tag{71}$$

E denoting their energy eigenvalue. A direct substitution in the stationary Schrödinger equation in the configuration space representation then reduces to, for any given value of x_2^c,

$$\left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x_2^2} + \frac{1}{2}m\omega_c^2\left(x_2 - x_2^c\right)^2 + \gamma x_2^c + \frac{1}{2}m\left(\frac{\gamma}{B}\right)^2\right\}\varphi_{E,x_2^c}(x_2) = E\,\varphi_{E,x_2^c}(x_2), \tag{72}$$

where one must also meet the condition $\varphi_{E,x_2^c}(x_2 = X_2) = 0$ which will imply a quantization rule for the energy values E. Note that indeed this operator possesses a self-adjoint extension for this choice of boundary conditions [11].

Introducing the notation,

$$u = \sqrt{\frac{2m\omega_c}{\hbar}}\left(x_2 - x_2^c\right), \qquad a = -\frac{1}{\hbar\omega_c}\left(E - \gamma\,x_2^c - \frac{1}{2}m\left(\frac{\gamma}{B}\right)^2\right), \tag{73}$$

the above eigenvalue equation becomes

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} - \left(\frac{1}{4}u^2 + a\right)\right)\varphi_{E,x_2^c}(u) = 0. \tag{74}$$

The general solution to this equation is a linear combination of the two parabolic cylinder functions $U(a, u)$ and $V(a, u)$ [12]. However since wave functions are required to vanish at $x_2 \to +\infty$, only the $U(a, u)$ branch is allowed[8]. Hence the solution is of the form,

$$\varphi_{E,x_2^c}(x_2) = N\left(E, x_2^c\right)U(a, u), \tag{75}$$

$N\left(E, x_2^c\right)$ being some normalization factor. Consequently the energy quantization condition is given by the boundary condition,

$$U\left(a, \sqrt{\frac{2m\omega_c}{\hbar}}\left(X_2 - x_2^c\right)\right) = 0. \tag{76}$$

Even though an explicit resolution of this condition requires a numerical analysis, given (74) it should be clear that this condition implies that the spectrum of a values belongs to a semi-infinite discrete set labelled as

$$a = -a_n\left(X_2 - x_2^c\right), \qquad n = 0, 1, 2, \ldots, \tag{77}$$

where each of the quantities $a_n\left(X_2 - x_2^c\right)$ is a continuous function of $\left(X_2 - x_2^c\right)$, while altogether they define a set of increasing values as n increases. Indeed, when multiplied by a factor (-1) and up to normalization factors, (74) is the Schrödinger wave equation for a harmonic oscillator with eigenvalue $(-a)$, of which the quadratic potential, namely $u^2/4$, is truncated for $u < \sqrt{2m\omega_c/\hbar}\left(X_2 - x_2^c\right)$. Since this potential with an infinite wall at $u = \sqrt{2m\omega_c/\hbar}\left(X_2 - x_2^c\right)$ is bounded below and unbounded above, its spectrum of standing waves and energy eigenvalues is certainly both bounded below and semi-infinite discrete, with growing eigenvalues $(-a)$ as the level index quantum number $n = 0, 1, 2, \ldots$ keeps increasing.

[8] Both functions $U(a, u)$ and $V(a, u)$ diverge as $u \to -\infty$, unless $a = -n - 1/2$ for $n = 0, 1, 2, \ldots$ in which case only $U(a, u)$ also vanishes in that limit, and in fact reduces [12] to $U(-n - 1/2, u) = 2^{-n/2}\,\mathrm{e}^{-u^2/4}H_n(u/\sqrt{2})$.

Note however that the energy quantization condition for a, hence its spectrum of values $a_n\left(X_2 - x_2^c\right)$, is independent of the parameter γ. For instance, since the function $U(a, u)$ vanishes in the limit $u \to -\infty$ only provided [12] $a = -n - 1/2$, one has

$$\lim_{X_2 \to -\infty} a_n\left(X_2 - x_2^c\right) = n + \tfrac{1}{2}, \qquad \lim_{x_2^c \to +\infty} a_n\left(X_2 - x_2^c\right) = n + \tfrac{1}{2}. \tag{78}$$

In conclusion, the configuration space wave functions of the energy eigenstates of the system are given in the form

$$\psi_{n,x_2^c}(x_1, x_2) = N\left(n, x_2^c\right) \mathrm{e}^{-\mathrm{i}\frac{m\omega_c}{\hbar} x_1 \left(x_2^c - \frac{1}{2} x_2 + \frac{\gamma}{m\omega_c^2}\right)} U\left(-a_n\left(X_2 - x_2^c\right), \sqrt{\frac{2m\omega_c}{\hbar}}\left(x_2 - x_2^c\right)\right), \tag{79}$$

$N\left(n, x_2^c\right)$ being a normalization constant to be determined, while the energy spectrum is

$$E\left(n, x_2^c\right) = \hbar\omega_c\, a_n\left(X_2 - x_2^c\right) + \gamma\, x_2^c + \frac{1}{2} m \left(\frac{\gamma}{B}\right)^2. \tag{80}$$

As compared to the results obtained in the plane in section 4, the only difference is the replacement by the quantities $a_n\left(X_2 - x_2^c\right)$ of the contributions in $(n + 1/2)$ of the left-handed chiral mode $\left(a_-, a_-^\dagger\right)$, while in the wave functions of these energy eigenstates the Hermite polynomial contribution multiplied by the Gaussian factor is replaced by that of the parabolic cylinder function. Furthermore there is no restriction whatsoever on the possible values for x_2^c, even though the particle remains confined to the $x_2 \geqslant X_2$ region. Yet the energy spectrum remains now bounded below.

Note that in the limit where the edge of the half-plane is pushed out again back to infinity, namely $X_2 \to -\infty$, the above results reduce smoothly back to those of section 4, as they should. However for any finite position of the edge at $x_2 = X_2$, even in the limit when $\gamma \to 0^+$, the energy spectrum remains non-degenerate since the values $a_n\left(X_2 - x_2^c\right)$ are functions of $\left(X_2 - x_2^c\right)$, hence of x_2^c and are independent of γ. In other words, the Landau level degeneracies of the ordinary Landau problem in the plane are lifted because of the interactions brought about by the edge—namely an infinite potential wall—at a finite position in the plane. Incidentally, the set of transformations in (54), identified first by considering the extra linear potential added to the ordinary Landau problem, proved essential in being able to construct the above explicit solution for the ordinary Landau problem in the half-plane.

Some features of the above complete solution may also be understood from the point of view of the classical trajectories in the half-plane, even in the presence or not of the constant force of strength γ. Given any value for $x_2^c > X_2$ however close to X_2, there always exist solutions of sufficiently small energy such that the radius of their circular motion about their (possibly moving, if $\gamma \neq 0$) magnetic centre remains less than the difference $\left(x_2^c - X_2 > 0\right)$, so that the particle then does not bounce off the wall at $x_2 = X_2$. Such trajectories are not distorted by the presence of the edge. At the quantum level because of the nonlocal nature of their wave function, such states display a slight deformation of their wave function, hence also of their energy value, but the less so the smaller is their energy and the larger is the value for x_2^c away from X_2. However as soon as the energy of the classical trajectory becomes large enough so that its radius becomes larger than $\left(x_2^c - X_2\right)$, the particle starts bouncing periodically off the wall at $x_2 = X_2$ in a series of elastic collisions, and is, in effect, set into motion—in case $\gamma \neq 0$ this extra motion is superposed on that of the magnetic centre already—along the x_1 axis in the positive direction. These trajectories are thus distorted, and even more so are the quantum states associated with such values of x_2^c and energy. Finally, even when x_2^c lies inside the 'forbidden' region, $x_2^c < X_2$, there do exist classical solutions of sufficiently large radius, namely energy, hence also quantum states of sufficiently large energy. But these states suffer the strongest distortion in wave function and energy values away from the equally spaced energy spectrum when the infinite wall at $x_2 = X_2$ is absent.

In terms of the effective harmonic potential contributing in the Schrödinger equation for $\varphi_{E,x_2^c}(x_2)$ in (72), namely

$$V_{\rm eff} = \frac{1}{2}m\omega_2^c\left(x_2 - x_2^c\right)^2 \qquad \text{for} \quad x_2 \geqslant X_2, \qquad V_{\rm eff} = +\infty \qquad \text{for} \quad x_2 < X_2, \tag{81}$$

which is thus truncated on the left-hand side for $x_2 < X_2$, the three typical situations discussed above correspond to when the minimum of that potential at $x_2 = x_2^c$ lies, respectively, well inside the region $x_2 > X_2$, or close to the edge, and finally inside the forbidden region $x_2 < X_2$. Solutions then correspond to standing waves inside this truncated harmonic well, which need to vanish at the infinite wall. As was noted previously, this is also the reason why the spectrum of $a_n\left(X_2 - x_2^c\right)$ values is always bounded below and discrete, as confirmed by a numerical analysis, and depends on x_2^c in such a manner that the total energy spectrum (80) remains bounded below however large and negative x_2^c may be.

6. Conclusions

This paper considered different variations on the same theme of the ordinary Landau problem. By first understanding why the choice of Landau gauge for the magnetic vector potential leads to non-countable and non-normalizable energy eigenstates whereas a solution in any other gauge produces an energy eigenbasis of countable and normalizable eigenstates, different canonical transformations of the phase space variables have been discussed enabling a straightforward resolution of different extensions of the Landau problem with a potential energy quadratic and linear in the plane cartesian coordinates.

As a matter of fact the main aim of this study was the explicit resolution solely through algebraic means of a singular extension of the Landau problem, namely when a potential energy only linear in the cartesian coordinates is introduced. Based on a specific canonical transformation, a clear separation of magnetic centre degrees of freedom and chiral rotating ones is achieved, allowing for a simple identification of the energy eigenspectrum and even of all its configuration space wave functions. The advantages of this approach are then finally brought to bear on the explicit and analytic solution of this singular Landau problem in the half-plane.

In order to come closer to an actual physical situation of physical interest in the quantum Hall context, whether the linear potential is related to a constant electric field or a gravitational well, the present study may be pursued by adding three more edges in the half-plane in order to build up a slab of finite extent, as a model for an actual experimental device for quantum Hall measurements. As a consequence, presumably the choice of variables which enabled the present solutions in the plane and in the half-plane will no longer be the most appropriate. Indeed, looking at the wave functions in (66) and (79), in that form it does not appear possible to enforce a vanishing wave function at two separate values of x_1. This is due to the fact that the plane wave component in x_1 of these wave functions derives from the eigenvalue value equation for the magnetic centre coordinate $\hat{x}_2^c$, which also contributes linearly to the Hamiltonian. However, since the two magnetic centre coordinates $\hat{x}_1^c$ and $\hat{x}_2^c$ do not commute, one cannot restrict both to be sharp, and by restricting the value of wave functions for specific values of x_1 certainly implies that energy eigenstates are no longer eigenstates of $\hat{x}_2^c$. In other words, like in the half-plane where one could no longer first diagonalize the Fock algebra and then produce the energy eigenspectrum, in a finite slab neither the Fock algebra nor the operator $\hat{x}_2^c$ may be diagonalized together with the Hamiltonian. From that point of view the diagonalization of the Hamiltonian in the form of (53), or possibly even in the Landau gauge rather than the symmetric gauge, certainly looks more appropriate. This issue deserves a dedicated study, which is likely to lead once again to parabolic cylinder functions in the x_2 direction, and ordinary trigonometric standing waves in the x_1 direction.

However, for what concerns the energy spectrum one should not expect a result that differs much qualitatively from that in (80). Namely besides a term analogous to the one linear in γx_2^c, there should be another contribution whose scale is set by the Landau problem itself, $\hbar\omega_c$. Furthermore, this latter contribution should remain independent of the coefficient setting the strength of the linear potential, γ. Consequently, the only effect of introducing this extra interaction energy in the system is to slightly tilt the spectrum of Landau levels.

To assess under which experimental conditions such effects may become observable, one needs to understand how much the energy spectrum—namely that of the density of states in conduction properties of such Hall probes—is a function of the geometry of such a finite slab as compared to the effects of a linear potential term, as may be induced by an electric field or the gravitational interaction. But whatever the finer details of these dependences turn out to be, it remains a fact that the factors setting the scales for these two types of effects are the Landau level gap, $\hbar\omega_c$, and the potential energy γx_2. This should allow for first-order estimates already.

Beyond the analysis considered here, another extension of potential interest is to include the spin $1/2$ degrees of freedom of the charged particle, as is indeed the case for electrons in actual quantum Hall experiments.

Finally, motivated by totally different considerations, it would also be interesting to reconsider the present singular Landau problem in the plane and the half-plane in the context of noncommutative quantum mechanics, namely for the Landau problem defined over the Moyal–Voros plane, using the techniques developed in [6, 13, 14] and having in mind to possibly identify some approach to experimentally set upper bounds on the noncommutativity parameter of noncommutative space(time) geometry.

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